

Counterexample for the 2-approximation of finding partitions of rectilinear polygons with minimum stabbing number

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Abstract. This paper presents a counterexample for the approximation algorithm proposed by Durocher and Mehrabi [1] for the general problem of finding a rectangular partition of a rectilinear polygon with minimum stabbing number.

1 Introduction

Given a rectilinear polygon P and a rectangular partition R of P , a segment is said to be **rectilinear** relative to P if it is parallel to one of P 's sides. Let s be a maximal rectilinear line segment inside P . The stabbing number of s relative to R is defined as the number of rectangles of R that s intersects. The stabbing number of R is the largest stabbing number of a maximal rectilinear line segment inside P . The Minimum Stabbing Rectangular Partition Problem (**MSRPP**) consists in finding a rectangular partition R of P having the smallest possible stabbing number. Figure 1 illustrates these definitions.

Variants of the problem arise from restricting the set of rectangular partitions that are considered to be valid. One of these variants is called the *conforming case*, in which every edge in the solution must be maximal, i.e., both of its endpoints must touch the border of the polygon. For this problem, in [1], Durocher et al. propose an integer programming model for the conforming case where there are exactly two edges (that can be in the solution) having each reflex vertex as endpoint. Thus, there are also precisely two variables associated to each reflex vertex.

In [1] a 2-approximation algorithm is presented for the conforming case of partitions of rectilinear polygons with minimum stabbing number. That approximation algorithm is based in a rounding of the variables. In the section named *Generalizing the Approximation Algorithm* of the article, it is stated that the algorithm could be extended for the general case using a formulation described informally and the same rounding rules used in the conforming case.

In this paper we show that the algorithm as described in [1] cannot give a 2-approximation for the general case of the (**MSRPP**). This is done by means of a counterexample to the referred algorithm.

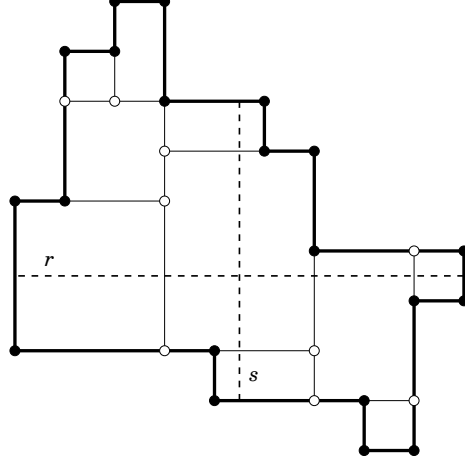


Fig. 1: A rectilinear polygon with a rectangular partition of stabbing number 4. The dashed lines represent maximal rectilinear line segments inside the polygon. Segment r has stabbing number 4 while segment s has stabbing number 3.

2 IP Models

The **MSRPP** can be modelled via integer programming in a number of different ways. In this section we present two such models for the general case of **MSRPP** in an attempt to formalize the description given in [1]. But first, we need some definitions.

Let P be a rectilinear polygon, input of the **MSRPP**. Define as V_r^P the set of **reflex vertices** of P , i.e., those having internal angles equal to $3\pi/2$. Let V_c^P be the set of vertices of P that are not reflex. Denote by $grid(P)$, the set of all maximal rectilinear line segments in the interior of P having a vertex in V_r^P as one of its endpoints. Let V_s^P be the set of points in the intersection of two segments in $grid(P)$. We refer to these points as **Steiner Vertices**. The points that are not in V_r^P or V_c^P and are in the intersection of a segment in $grid(P)$ and the border of P compose the set V_b^P . Denote by V^P the set resulting from the union of all the point sets defined before, i.e., $V^P = V_r^P \cup V_c^P \cup V_s^P \cup V_b^P$.

Define E_h^P as the set of line segments in the border of P having only two points in V^P which are its extremities. Any fragment of a segment in $grid(P)$ containing exactly two vertices in V^P is called an **internal edge**. The set of all internal edges is E_i^P and the set of all edges in P is $E^P = E_h^P \cup E_i^P$. A subset E'^P of E^P defines a *knee* in a vertex $u \in V_s^P \cup V_r^P$ if exactly two edges in E'^P have u as an endpoint and these edges are orthogonal. A subset E'^P of E^P is said to define an *island* in a vertex $u \in V_r^P$ if only one edge of E'^P have u as an endpoint. At last, if ua and ub are two edges in E^P having a common endpoint u , we denote the angle between ua and ub by $\theta(ua, ub)$.

Now, we can formalize the model described in [1] as follows:

$$(RPST) \quad z = \min k \quad (1)$$

$$\text{subject to} \quad x_{ua} + x_{ub} \geq 1, \quad \forall u \in V_r^P \wedge ua, ub \in E_i^P, \quad (2)$$

$$x_{ua} + x_{ub} - x_{uc} \geq 0, \quad \forall u \in V_s^P, \forall ua, ub, uc \in E_i^P \quad (3)$$

with $\theta(ua, ub) = \pi/2$,

$$\sum_{\substack{uv \in E_i^P \\ uv \cap s \neq \emptyset}} x_{uv} \leq k - 1, \quad \forall s \in L, \quad (4)$$

$$x_{uv} \in \mathbb{B} \quad \forall uv \in E_i^P, \quad (5)$$

$$k \in \mathbb{Z}. \quad (6)$$

In the model above, we have one binary variable x_{uv} for each internal edge uv in P which is set to 1 if and only if the corresponding edge is in the rectangular partition of P . Constraints (2) ensure that the solution does not contain a knee in a reflex vertex. Inequalities (3) impose that the solution does not form a knee or an island in a Steiner vertex. Inequalities (4) relate the x variables with variable k , which represents the stabbing number of the solution. As a consequence, the objective function (1) is to minimize k . Finally, (5) and (6) are integrality restrictions for the variables. Figure 2 shows an instance of the **MSRPP** (called **random-20-17**) with 62 internal edges and their corresponding variables.

random-20-17

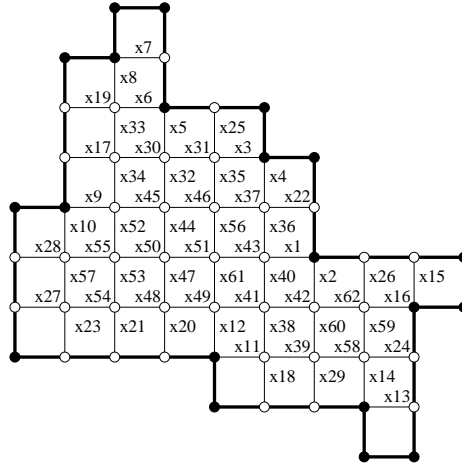


Fig. 2: Instance **random-20-17** with 62 internal edges and the corresponding variables.

As stated before, the (*RPST*) model above is not the only model for the problem and next we show another way of modelling it. However, to guarantee the correctness of the model we must first prove a property of optimal solutions for the **MSRPP**. The following proposition is a generalization of *Observation 1* in [1].

Proposition 1. *Any rectilinear polygon P has an optimal rectangular partition R in which every maximal segment of R has at least one reflex vertex of P as an endpoint.*

Proof. Let R be a rectangular partition of a rectilinear polygon P . Let e be a maximal segment in R having a and b as its endpoints. Suppose neither a nor b are reflex vertices. Since e is maximal and R is a rectangular partition, both endpoints of e must lie in segments perpendicular to e .

Now, since R is a rectangular partition, e define two minimal rectangles (each one possibly containing other rectangles) having e as one of its sides, let us denote them by r_1 and r_2 . There are three cases to consider.

The first case consists of r_1 and r_2 been empty rectangles, i.e., neither r_1 nor r_2 contain other rectangles. Therefore, the removal of e unite these rectangles, composing a single rectangle. Therefore, $R \setminus e$ is still a rectangular partition. It is clear that removing a segment cannot increase the stabbing number of the solution. Thus, if R is an optimal solution, so is $R \setminus e$.

The second case to consider is when only one of r_1 or r_2 contains other rectangles. Suppose without loss of generality that r_1 is the one containing other rectangles. Now, we can drag e towards r_1 , shrinking any segment with an endpoint in e , until e meets a reflex vertex or the border of P . In the latter case, e is merged to the border of P . It is easy to see that the result of this dragging operation is also a rectangular partition besides, the only stabbing segments affected by this operation are the ones parallel to e and their stabbing number cannot increase. Therefore, as R is optimal, so must be the new solution.

At last, we must consider the case where both r_1 and r_2 contain other rectangles. Suppose without loss of generality that the number of segments in r_1 having an endpoint in e (thus, perpendicular to it) is greater or equal than the number of segments with these characteristics in r_2 . Then, again, we can drag e towards r_1 , shrinking any segment with an endpoint in e , until e meets either a segment parallel to e or a reflex vertex or the border of P . If a parallel segment is met, e is merged to it and the process is repeated until a reflex vertex or the border of P is met. In case the border of P is met, e ceases to exist together with the segments in the space between e and the border. Once again, the dragging operation results in a rectangular partition of P and the only stabbing segments affected by this operation are parallel to e . But, as the number of segments in r_1 is greater or equal than the number of segments in r_2 , one can see that the stabbing number of the new rectangular partition cannot be greater than that of R .

Ergo, there is always an optimal rectangular partition where every maximal segment has at least one reflex vertex of P as an endpoint. \square

In the next model, given the same definitions as before, we consider the set E_e^P of rectilinear segments uv where $u \in V_r^P$ and $v \in V^P$. Notice that a segment of E_e^P can be comprised of several consecutive segments of E_i^P . Hence, we call E_e^P the **extended edge** set. In the formulation below, we have a variable x_{uv} for each edge in E_e^P and from Proposition 1 it is easy to notice that this set of variables is sufficient to provide optimal rectangular partitions.

$$(RPST2) \quad z = \min k \quad (7)$$

subject to

$$\sum_{ua \in E_e^P} x_{ua} \geq 1, \quad \forall u \in V_r^P \quad (8)$$

$$x_{ab} + x_{uv} \leq 1, \quad \forall ab, uv : ab \cap uv \neq \emptyset \wedge \\ \wedge ab \cap uv \neq a, b, u \text{ or } v \quad (9)$$

$$\sum_{\substack{\theta(uv, ab) = \pi/2 \wedge \\ b \in uv \wedge b \neq u \wedge b \neq v}} x_{uv} - x_{ab} \geq 0, \quad \forall a \in V_r^P, b \in V_s^P \quad (10)$$

$$\sum_{uv \in E_e^P : uv \cap s \neq \emptyset} x_{uv} \leq k - 1, \quad \forall s \in L \quad (11)$$

$$x_{uv} \in \mathbb{B} \quad \forall uv \in E_e^P. \quad (12)$$

$$k \in \mathbb{Z} \quad (13)$$

In this model, inequalities (8) guarantee that the solution does not contain a knee in a reflex vertex. Constraints (9) enforce planarity (two segments of the partition can only intersect at their extremes). Constraints (10) prevent the existence of knees and islands in a Steiner vertex. Finally, (11) are the stabbing constraints and (12) and (13) are integrality constraints. Figure 3 shows instance **random-20-17** with 42 internal edges and the corresponding variables.

3 The Counterexample

Before discussing the counterexample, we first present the rounding scheme proposed in [1] for the conforming case. Once the optimum of the linear relaxation is computed, the rules for rounding variables in the conforming case are really simple: a variable corresponding to a horizontal segment is rounded down to zero if its value is smaller than or equal to 0.5 and is rounded up to one if its value is greater than 0.5. A variable corresponding to a vertical segment is rounded down to zero if its value is smaller than 0.5 and is rounded up to one if its value is greater than or equal to 0.5.

In the *Generalizing the Approximation Algorithm* section of [1], a model for the general (non-conforming) case is described informally. From the discussion, apparently such model is equivalent to the (*RPST*) formulation given in Section 2. According to the authors, the same rounding rules used in the conforming case provide a 2-approximation for the general case.

random-20-17

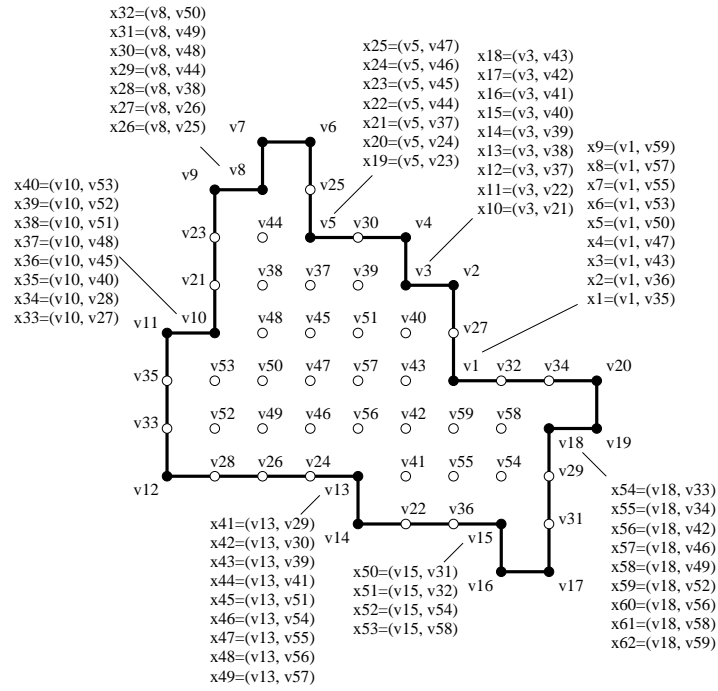


Fig. 3: Instance random-20-17 with its extended edges and corresponding variables.

The rounding rules do not mention what should be done for Steiner vertices, and no guarantee is given that applying them directly in these situations will avoid the formation of a knee or an island. In fact, the instance displayed in Figure 4 shows that this cannot always be done without sacrificing feasibility. In this figure, the optimal values of the variables corresponding to edges incident to Steiner vertex **v38** (see Figure 3) after solving the linear relaxation associated to instance **random-20-17** are given. As only the variable corresponding to one vertical edge incident to that vertex has value equal to 0.5 and the other three are smaller than 0.5, rounding according to that rule would result in an island at **v38**. Therefore, the set of edges obtaining after rounding does not form a rectangular partition.

random-20-17

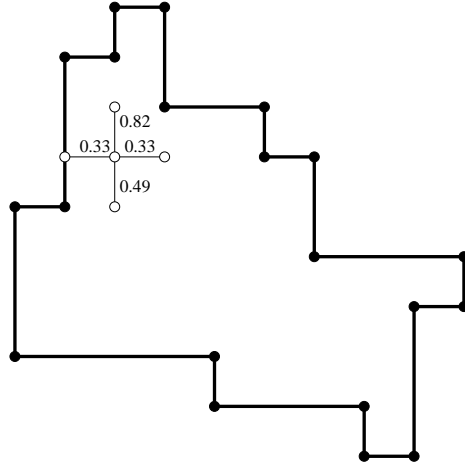


Fig. 4: Values of variables corresponding to edges incident to a Steiner vertex after solving linear relaxation. The values are rounded with two digits after the decimal point.

It is however possible that we misinterpreted the model the authors were thinking of (although there is evidence in contrary) and the idea is actually to define variables corresponding to all edges having a reflex vertex as one of its endpoints. If so, the formulation would look like (*RPST2*) model in the previous section. In this alternative formulation, rounding the variables using that rule does not cause the same problem as before since every variable correspond to an edge having a reflex vertex as endpoint.

Contrary to what happens in the conforming case, however, the reflex vertices here have more than two incident edges. Therefore, it is possible that the solution of the linear relaxation result in values smaller than 0.5 for all the variables

corresponding to the edges incident to a certain reflex vertex. Thus, the rounding of such solution would result in a partition having a knee in a reflex vertex.

The situation described above occurs in practice with instance `random-20-17`, as shown in Figure 5. Consider the edges incident to vertex `v3`. All the associated variables incident to this vertex have value smaller than 0.5. As consequence, they will be rounded to zero, resulting in the formation of a knee at `v3` and, therefore, in an infeasible solution.

random-20-17

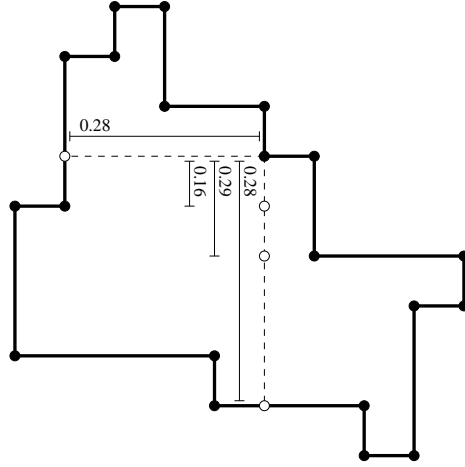


Fig. 5: Values of variables corresponding to edges incident to a reflex vertex after solving linear relaxation. Variables with value zero are omitted. The values are rounded with two digits after the decimal point.

4 Conclusion

From the counterexample presented in Section 3, we conclude that it remains open whether a 2-approximation for the **MSRPP** in the general case exists. It is, however, noteworthy that many other contributions are presented in [1] and none of them are diminished by this counterexample.

References

1. Stephane Durocher and Saeed Mehrabi. Computing partitions of rectilinear polygons with minimum stabbing number. In Joachim Gudmundsson, Julián Mestre, and Taso Viglas, editors, *Computing and Combinatorics*, volume 7434 of *Lecture Notes in Computer Science*, pages 228–239. Springer Berlin Heidelberg, 2012.

Appendix

File name: random-20-17.rect

Model: RPST

Vertex number: 59

Edge number: 62

Reading Problem stab

Problem Statistics

231 (0 spare) rows
63 (0 spare) structural columns
752 (0 spare) non-zero elements

Global Statistics

63 entities 0 sets 0 set members

Minimizing MILP stab

Original problem has:

231 rows 63 cols 752 elements 63 globals

Will try to keep branch and bound tree memory usage below 6.1Gb

Its	Obj Value	S	Ninf	Nneg	Sum Dual	Inf	Time
0	.000000	D	24	0	.000000		0
87	2.411765	D	0	0	.000000		0

Optimal solution found

*** Search unfinished *** Time: 0 Nodes: 0

Number of integer feasible solutions found is 0

Best bound is 2.411765

Solution:

x1 = 0.568627	x2 = 0.431373	x3 = 0.274510	x4 = 0.725490
x5 = 0.470588	x6 = 0.529412	x7 = 0.000000	x8 = 1.000000
x9 = 0.555556	x10 = 0.444444	x11 = 0.686275	x12 = 0.313725
x13 = 0.705882	x14 = 0.294118	x15 = 0.294118	x16 = 0.705882
x17 = 0.326797	x18 = 0.686275	x19 = 0.183007	x20 = -0.000000
x21 = -0.000000	x22 = 0.156863	x23 = 0.098039	x24 = -0.000000
x25 = 0.124183	x26 = -0.000000	x27 = 0.000000	x28 = 0.346405
x29 = 0.431373	x30 = 0.326797	x31 = 0.326797	x32 = 0.143791
x33 = 0.816993	x34 = 0.490196	x35 = 0.052288	x36 = 0.568627
x37 = 0.156863	x38 = 0.274510	x39 = 0.411765	x40 = 0.274510
x41 = 0.000000	x42 = 0.274510	x43 = 0.294118	x44 = 0.000000
x45 = 0.209150	x46 = 0.209150	x47 = 0.000000	x48 = -0.000000
x49 = 0.000000	x50 = 0.346405	x51 = 0.346405	x52 = 0.346405
x53 = 0.000000	x54 = 0.000000	x55 = 0.346405	x56 = 0.052288
x57 = 0.098039	x58 = -0.000000	x59 = 0.294118	x60 = 0.431373
x61 = 0.313725	x62 = 0.705882	x63 = 2.411765	

File name: random-20-17.rect

Model: RPST2

Vertex number: 59

Edge number: 62

Reading Problem stab

Problem Statistics

336 (0 spare) rows
63 (0 spare) structural columns
996 (0 spare) non-zero elements

Global Statistics

63 entities 0 sets 0 set members

Minimizing MILP stab

Original problem has:

336 rows 63 cols 996 elements 63 globals

Crash basis containing 13 structural columns created

Its	Obj Value	S	Ninf	Nneg	Sum Inf	Time
0	.000000	D	1	0	24.000000	0
82	2.413793	D	0	0	.000000	0

Optimal solution found

*** Search unfinished *** Time: 0

Number of integer feasible solutions found is 0

Best bound is 2.413793

Solution:

x1 = 0.293103 x2 = 0.431034 x3 = 0.275862 x4 = -0.000000
x5 = -0.000000 x6 = -0.000000 x7 = -0.000000 x8 = -0.000000
x9 = -0.000000 x10 = 0.275862 x11 = 0.275862 x12 = -0.000000
x13 = -0.000000 x14 = -0.000000 x15 = 0.155172 x16 = -0.000000
x17 = -0.000000 x18 = 0.293103 x19 = 0.241379 x20 = -0.000000
x21 = 0.275862 x22 = 0.293103 x23 = 0.051724 x24 = -0.000000
x25 = 0.137931 x26 = -0.000000 x27 = -0.000000 x28 = 0.275862
x29 = 0.241379 x30 = 0.189655 x31 = -0.000000 x32 = 0.293103
x33 = 0.155172 x34 = 0.103448 x35 = -0.000000 x36 = 0.137931
x37 = 0.293103 x38 = 0.017241 x39 = 0.000000 x40 = 0.293103
x41 = 0.000000 x42 = 0.017241 x43 = -0.000000 x44 = 0.275862
x45 = -0.000000 x46 = 0.293103 x47 = 0.120690 x48 = 0.000000
x49 = 0.293103 x50 = 0.706897 x51 = -0.000000 x52 = -0.000000
x53 = 0.293103 x54 = -0.000000 x55 = 0.293103 x56 = 0.275862
x57 = 0.000000 x58 = -0.000000 x59 = -0.000000 x60 = -0.000000
x61 = 0.000000 x62 = 0.431034 x63 = 2.413793